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# A Novel Quantification of 3D Directional Spread from Small-Scale Fading Analysis

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**Abstract** – The directional dispersion of multipath energy arriving at the receiver is an important quantity, as increasing it leads to lower correlation between spatial diversity elements. The ‘RMS angular spread’ metric, commonly used to measure this, is only applicable for relatively small angles in the 2-D plane. The quantification of the 3-D directional spread is not straightforward, and must be justified analytically by showing its relation to spatial selectivity, and hence spatial correlation. In this paper, an analysis of the second order moments of spatial fading has been used to define a spatial covariance matrix, which fully describes the second order moments of the directional energy distribution. An eigenvalue decomposition of this matrix has been used to propose a concise quantification of directional spread that is applicable for arbitrary distribution of multipath rays or clusters in 3-D directions. The effect of array orientation has also been investigated.

## I. INTRODUCTION

The duality between directional dispersion of multipath energy and spatial selectivity of the received power envelope is well known [1][2]. Directional dispersion at either terminals of a MIMO communication link is a critical quantity, as increasing it leads to increased fluctuations in received power over space, and consequently, lower correlation between spatial diversity elements. Measurement of this directional dispersion is needed in the analysis of multiple antenna systems and development of physical channel models, and forms a crucial aspect of radio propagation characterization. The standard deviation or the root-mean-square (RMS) spread of the azimuth angular energy distribution is a commonly used measure, and has been shown to be the quantity that determines spatial correlation, as opposed to the specific angular distribution function [3]. Although the RMS measure is applicable only to angles in a given 2-D plane, it has been widely used as spatial fading statistics such as coherence distances, level-crossing rates and fade durations have been traditionally analysed using simplistic azimuthal propagation models [4]. However, the use of this metric becomes cumbersome when 3-D directions of arrival are considered, as it must be stated separately for the azimuth and the elevation domains. In addition, since the power azimuth spectrum is a periodic function, the RMS measure is unsuitable for large angular separations, and must be applied separately for each cluster when there are multiple well-spaced clusters [5].

An alternative metric is a quantity called ‘multipath richness’. Proposed metrics for multipath richness include the number of independent resolvable paths propagating from the transmitter to the receiver [6], the effective degrees of freedom (EDOF) or the slope of capacity versus signal-to-noise ratio (SNR) at one value of SNR [7][8], the relative sum of singular

values of the channel response matrix [9] and the cumulative sum of the log of normalized eigenvalues [10]. These metrics are useful for gauging the fading statistics; but since they are calculated from the MIMO channel response, they have a dependency on the radiation patterns and placements of the antenna elements. However, directional dispersion or directional spread is a property of radio propagation and should ideally be calculated from the directional energy distribution. The use of  $K$ -factor as a measure of directional spread is also incorrect. Although the presence of a dominant path can limit the directional energy spread, the contrary is not true as lower  $K$ -factors do not necessarily imply large directional spreads.

In [11], a definition for 3-D directional spread was proposed by following the analogy of the delay and Doppler spread. The relationship between the direction-constrained Doppler frequency spread and the directional distribution of propagation was explored, and the maximum spatial Doppler spread was found to be related to directional spread via an inequality. The maximum Doppler spread and the direction spread proposed here coincided with two of the three shape factors that were proposed for characterizing arbitrary azimuth energy distributions in [12].

In this paper, we aim to provide a generic analytical framework for the analysis and quantification of directional dispersion. The duality between directional dispersion and spatial selectivity has been exploited to propose a measure of direction spread that is applicable for arbitrary distribution of multipath rays or clusters in 3-D directions. The analysis uses the second-order moment of the gradient of the spatial fading function (similar to [12]), and constructs a spatial covariance matrix  $\mathbf{R}$  that fully describes the second order moment of the directional energy distribution. Unlike previous work, the analysis here allows for the use of an eigenvalue-based analysis, which provides a physically intuitive interpretation of directional dispersion and its characterization. The trace of  $\mathbf{R}$  is mathematically equivalent to the metrics proposed previously in [11][12] and [13]. In addition, we find that the determinant of  $\mathbf{R}$  is an important characteristic of directional dispersion. The joint use of the  $\text{tr}(\mathbf{R})$  and  $\det(\mathbf{R})$  is proposed as a concise and comprehensive quantification of the 3-D directional spread.

## II. THEORETICAL ANALYSIS

### A. Directional Derivatives of the Fading Function

Small-scale fading or spatial selectivity refers to the rapid fluctuation of received power over sub-wavelength distances

in space that is produced by scattered electromagnetic waves arriving from diverse directions. The statistics of small-scale fading is critical to inter-element correlation and ultimately the information theoretical capacity of Multiple-Input Multiple-Output (MIMO) channels employing space-diversity arrays. The rate of fluctuation of the spatially fading stationary process is given by the first order spatial derivative of the received voltage, received power or received envelope. The analysis in this paper has been done for received power, which is denoted for a point in space  $\mathbf{s} = [x, y, z]$  as  $f(x, y, z)$ . The vector representing the first order partial derivatives is given by  $\nabla f = [f_x, f_y, f_z]$ . The rate of variation of  $f(x, y, z)$  in a given direction  $\mathbf{u}$  is given by the three-dimensional directional derivative of  $f(x, y, z)$ , and is denoted as  $m_{\mathbf{u}}$ . The directional unit vector  $\hat{\mathbf{u}}$  is defined in Cartesian and spherical co-ordinates as

$$\hat{\mathbf{u}} = \begin{bmatrix} u_x \\ u_y \\ u_z \end{bmatrix} = \begin{bmatrix} \sin \theta \cos \phi \\ \sin \theta \sin \phi \\ \cos \theta \end{bmatrix} \quad (1)$$

where  $(\theta, \phi)$  represent the spherical co-ordinate angles.  $m_{\mathbf{u}}$  is given by (2).

$$\begin{aligned} m_{\mathbf{u}} &= \nabla_{\mathbf{u}} f = \nabla f \cdot \frac{\mathbf{u}}{|\mathbf{u}|} \\ &= \lim_{l \rightarrow 0} \frac{f(\mathbf{x} + l\hat{\mathbf{u}}) - f(\mathbf{x})}{l} \\ &= \frac{\partial f}{\partial x} u_x + \frac{\partial f}{\partial y} u_y + \frac{\partial f}{\partial z} u_z \\ &= f_x u_x + f_y u_y + f_z u_z \end{aligned} \quad (2)$$

The mean of  $m_{\mathbf{u}}$  taken over many realisations of independent multipath phases must be zero. The mean square or the second order moment of  $m_{\mathbf{u}}$ , denoted as  $\sigma_m^2(\hat{\mathbf{u}})$ , is the statistic that gives a measure of the fading rate in the direction  $\mathbf{u}$  as is given by

$$\begin{aligned} \sigma_m^2(\hat{\mathbf{u}}) &= E[m_{\mathbf{u}}^2] = E[f_x^2] u_x^2 + E[f_y^2] u_y^2 + E[f_z^2] u_z^2 + E[f_x f_y] u_x u_y + \\ &\quad + E[f_y f_z] u_y u_z + E[f_z f_x] u_z u_x \end{aligned} \quad (3)$$

where the second order moments of the partial derivatives are denoted as follows:

$$\begin{aligned} R_{xx} &= E[f_x^2] & R_{yy} &= E[f_y^2] & R_{zz} &= E[f_z^2] \\ R_{xy} &= E[f_x f_y] & R_{yz} &= E[f_y f_z] & R_{zx} &= E[f_z f_x] \end{aligned} \quad (4)$$

The  $\sigma_m^2(\hat{\mathbf{u}})$  describes the second order statistics of small-scale fading at any point in space as a function of direction emanating from the point. In order to estimate  $\sigma_m^2(\hat{\mathbf{u}})$  efficiently, simple expressions are needed for  $R_{xx}$ ,  $R_{yy}$ ,  $R_{zz}$ ,  $R_{xy}$ ,  $R_{yz}$  and  $R_{zx}$ . Expressions for  $(f_x, f_y, f_z)$  and their covariances are derived in Section II.B.

### B. Covariance of Partial Derivatives

The power of a spatially fading stationary wave is given by  $f = |h|^2$ , where  $h$  is the normalised channel response given by the summation of  $N_s$  multipath waves. The assumption that electromagnetic radiation arrives at the receiver in planar

wavefronts will be used. Denoting the direction of arrival as a vector  $(\mathbf{d} = [d_x, d_y, d_z])$ ,  $h$  is given by

$$h = \sum_{s=1}^{N_s} A_s \exp(i(\hat{\mathbf{d}}_s \cdot \mathbf{s} + \chi_s)) = \sum_{s=1}^{N_s} A_s \exp(i\psi_s) \quad (5)$$

where  $\psi_s$  denotes the overall phase of the  $s$ th multipath ray and  $A_s^2$  is power of each ray after normalization so that the total power equals unity. The expressions for the partial derivatives are shown for the case of  $f_x$ .

$$f_x = \frac{\partial |h|^2}{\partial x} = h^* \frac{\partial h}{\partial x} + h \frac{\partial h^*}{\partial x} = 2 \operatorname{Re} \left\{ \frac{\partial h}{\partial x} h^* \right\} \quad (6)$$

$$\frac{\partial h}{\partial x} = \sum_{s=1}^{N_s} A_s i d_{x_s} \exp(i\psi_s) \quad (7)$$

Using (5), (6) and (7), and ignoring the multiplication factor of 2 in (6),  $f_x$  simplifies to a summation of a large number of independent and identically distributed (i.i.d) random variables, as shown in (8). The central limit theorem dictates that  $f_x, f_y$  and  $f_z$  are gaussian distributed.

$$\begin{aligned} f_x &= \operatorname{Re} \left\{ \sum_{s=1}^{N_s} A_s i d_{x_s} \exp(i\psi_s) \cdot \sum_{k=1}^{N_s} A_k \exp(-i\psi_k) \right\} \\ &= \sum_{s=1}^{N_s} \sum_{k=1}^{N_s} A_s A_k d_{x_s} \sin(\psi_s - \psi_k) \\ &= \sum_{s=1}^{N_s} \sum_{k=1}^s A_s A_k (d_{x_s} - d_{x_k}) \sin(\psi_s - \psi_k) \end{aligned} \quad (8)$$

A derivation of the general expression for the second order moment of the partial derivatives, as given by (9), is shown in Appendix A.

$$R_{xy} = \frac{1}{2} \sum_{s=1}^{N_s} A_s^2 (d_{x_s} - \bar{d}_x)(d_{y_s} - \bar{d}_y) \quad (9)$$

More specifically, the variance in any given direction is given by (10).

$$R_{xx} = \frac{1}{2} \sum_{s=1}^{N_s} A_s^2 (d_{x_s} - \bar{d}_x)^2 \quad (10)$$

In (9) and (10),  $\bar{d}_x$  and  $\bar{d}_y$  are components of  $\bar{\mathbf{d}}$  where

$$\bar{\mathbf{d}} = \sum_{s=1}^{N_s} A_s^2 \mathbf{d}_s = \sum_{s=1}^{N_s} A_s^2 (d_{x_s} \hat{\mathbf{i}} + d_{y_s} \hat{\mathbf{j}} + d_{z_s} \hat{\mathbf{k}}) = \bar{d}_x \hat{\mathbf{i}} + \bar{d}_y \hat{\mathbf{j}} + \bar{d}_z \hat{\mathbf{k}} \quad (11)$$

Equations (9) and (10) take the form of general expressions for cross-covariance and auto-covariance respectively, except that the directional components here are weighted by the powers of the multipath components instead of probabilities. There is no phase dependency as they are calculated only from the multipath powers and directions. Since the powers and directional vectors are normalised to unity, the upper bound of any of the second order moments is equal to 0.5.

### C. Quantifying Directional Spread

Since the second order moments  $R_{xx}$ ,  $R_{yy}$ ,  $R_{zz}$ ,  $R_{xy}$ ,  $R_{yz}$  and  $R_{zx}$  are directly related to the separation of the directional vector components,  $\sigma_m^2(\hat{\mathbf{u}})$  can be expected to increase with the overall directional spread.  $\sigma_m^2(\hat{\mathbf{u}})$  can be calculated directly for any given multipath energy distribution using the results given in (9) and (10). However,  $\sigma_m^2(\hat{\mathbf{u}})$  has by definition a dependency on direction and for conciseness, a scalar quantity that includes the fading statistics over all spherical directions might be preferred instead. In this section a simple quantification of directional spread is proposed using an eigenvalue analysis of  $(\nabla f)(\nabla f)^T$ .

For a generic directional distribution of multipath power, the distribution of  $\nabla f$  over the 3 euclidean co-ordinates as calculated for independent phase realisations of the multipath components takes the shape of an ellipsoid. The eigenvalue decomposition of the covariance matrix of  $\nabla f$  is given by

$$\mathbf{R} = E[(\nabla f)(\nabla f)^T] = \begin{bmatrix} R_{xx} & R_{xy} & R_{zx} \\ R_{xy} & R_{yy} & R_{yz} \\ R_{zx} & R_{yz} & R_{zz} \end{bmatrix} = \mathbf{P}\mathbf{\Lambda}\mathbf{P}^T \quad (12)$$

where  $\mathbf{\Lambda} = \text{diag}\{\lambda_1 \lambda_2 \lambda_3\}$  and  $\mathbf{P} = [\mathbf{e}_1 \mathbf{e}_2 \mathbf{e}_3]$ . The constituent basis vectors  $\mathbf{e}_i$  give the principle directions of the ellipsoid. For  $\lambda_1 > \lambda_2 > \lambda_3$ , the vector  $\mathbf{e}_1$  gives the direction of maximum stretch of the distribution of  $\nabla f$  in the Euclidean space. The sum of eigenvalues of a correlation matrix is equivalent to its trace, as shown in (13).

$$\text{trace}(\mathbf{R}) = \lambda_1 + \lambda_2 + \lambda_3 = R_{xx} + R_{yy} + R_{zz} \quad (13)$$

Now, consider the mean of  $\sigma_m^2(\hat{\mathbf{u}})$  over all 3-D directions  $\mathbf{u}$ . Random directional unit vectors that give uniformly distributed points on the surface of the sphere can be generated using numerous methods based on [14], where  $u_x$ ,  $u_y$  and  $u_z$  are independent random variables uniformly distributed in the range  $[-1, 1]$ . Denoting their probability density functions as  $p(u_x)$ ,  $p(u_y)$  and  $p(u_z)$ , the mean is given by (14).

$$\begin{aligned} E[\sigma_m^2(\hat{\mathbf{u}})] &= R_{xx}E[u_x^2] + R_{yy}E[u_y^2] + R_{zz}E[u_z^2] + 2R_{xy}E[u_x u_y] + \\ &\quad + 2R_{yz}E[u_y u_z] + 2R_{zx}E[u_z u_x] \\ &= R_{xx} \int_{-1}^1 u_x^2 p(u_x) du_x + R_{yy} \int_{-1}^1 u_y^2 p(u_y) du_y + R_{zz} \int_{-1}^1 u_z^2 p(u_z) du_z \\ &= \frac{1}{3}(R_{xx} + R_{yy} + R_{zz}) \end{aligned} \quad (14)$$

Thus,  $\text{tr}(\mathbf{R})$  gives a measure of the second order fading statistics averaged over all spherical directions and can be rewritten as

$$\begin{aligned} \text{tr}(\mathbf{R}) &= R_{xx} + R_{yy} + R_{zz} \\ &= \frac{1}{2} \sum_s^{N_s} A_s^2 [(d_x - \bar{d}_x)^2 + (d_y - \bar{d}_y)^2 + (d_z - \bar{d}_z)^2] \\ &= \frac{1}{2} \sum_s^{N_s} A_s^2 D_s^2 \end{aligned} \quad (15)$$

where  $D_s$  represents the Euclidean distance between directional unit vectors  $\mathbf{d}_s$  and the mean directional vector  $\bar{\mathbf{d}}$ . Thus  $\text{tr}(\mathbf{R})$  can be calculated directly using only the powers and 3-D directions of multipath components. The theoretical upper bound used here for  $\text{tr}(\mathbf{R})$  is equal to 0.5. The  $\text{tr}(\mathbf{R})$  is mathematically equivalent to the ‘‘multipath component separation’’ proposed in [13], and the direction spread metric proposed in [11].

However, higher  $\text{tr}(\mathbf{R})$  or RMS angular spread do not automatically imply lower correlation between space-diversity antennas as there is a dependency on array orientation as well. The level of variation in second order fading statistics with direction is modelled by the determinant of the correlation matrix, which is equivalent to the product of the eigenvalues, as given by (16).

$$\det(\mathbf{R}) = \lambda_1 \lambda_2 \lambda_3 \quad (16)$$

The  $\det(\mathbf{R})$  gives a measure of invariance of the second order fading statistics with direction. As  $\mathbf{R}$  becomes more ill-conditioned,  $\det(\mathbf{R})$  decreases and the variation in  $\sigma_m^2(\hat{\mathbf{u}})$  with  $\mathbf{u}$  increases. The theoretical upper bound of  $\det(\mathbf{R})$  is achieved when the eigenvalues are equal and their summation equals 0.5, such that  $\lambda_1 \lambda_2 \lambda_3 = (0.5/3)^3 = 0.0046$ . This occurs when the distribution of multipath power is perfectly isotropic and variation in  $\sigma_m^2(\mathbf{u})$  with  $\mathbf{u}$  is zero. The  $\text{tr}(\mathbf{R})$  also achieves its upper bound in this scenario.

For multipath profiles comprising of only one directional cluster, both  $\text{tr}(\mathbf{R})$  and  $\det(\mathbf{R})$  can be expected to improve with the directional spread. The importance of  $\det(\mathbf{R})$  is highlighted by the case where the multipath distribution comprises only two rays approaching from directly opposite directions. Here,  $\text{tr}(\mathbf{R})$  and RMS angular spread achieve their upper bounds, but  $\det(\mathbf{R})$  is equal to zero as the variation in  $\sigma_m^2(\mathbf{u})$  with  $\mathbf{u}$  is greatest.  $\sigma_m^2(\hat{\mathbf{u}})$  is maximum when  $\mathbf{u}$  is parallel to the line of the opposite rays, but zero when  $\mathbf{u}$  is perpendicular to the rays implying perfectly correlated antennas for this orientation of a linear array. Hence,  $\text{tr}(\mathbf{R})$  and  $\det(\mathbf{R})$  are both required for measuring the directional dispersion, as they characterize the average and the variation of  $\sigma_m^2(\hat{\mathbf{u}})$  over the 3D directional space.

### III. SIMULATION RESULTS

A diagrammatical demonstration of the theory introduced in Section II is presented in this section. Generic directional distributions comprising of either one or two clusters of multipath components were constructed, as shown in Figure 1. For simplicity, the cluster energies were modelled to be uniformly distributed in the range of  $[-\Gamma/2, \Gamma/2]$  in the azimuth and  $[-\Delta/2, \Delta/2]$  in elevation, with clusters mean directions contained within the azimuth (or x-y) plane. For the two cluster distribution, the angular separation between the mean directions of the clusters was denoted as  $\beta$ .

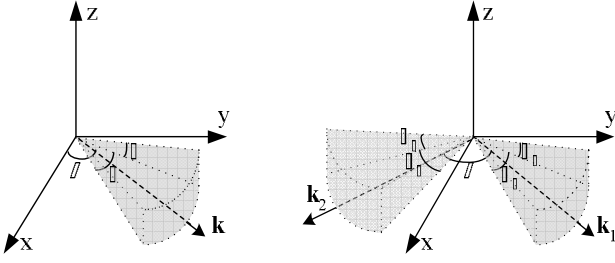


Figure 1: Multipath distributions comprising of one or two clusters and their parameters. The cluster means  $\mathbf{k}_1$  and  $\mathbf{k}_2$  are contained within the x-y plane.

1) *Second order moments of fading*: A validation for the derived expressions given in (9) and (10) was obtained by comparing them with their direct estimates, which were calculated using (4). The direct estimates were calculated from a large number of realisations of the partial derivatives ( $f_x, f_y, f_z$ ), which were obtained using distinct and independent multipath phase profiles. The multipath component phases within each profile were assumed to be random and uniformly distributed in  $[0, 2\pi)$ , as was suggested in [15]. A close match was found between the estimates and the derived equations, as shown in Figure 2.

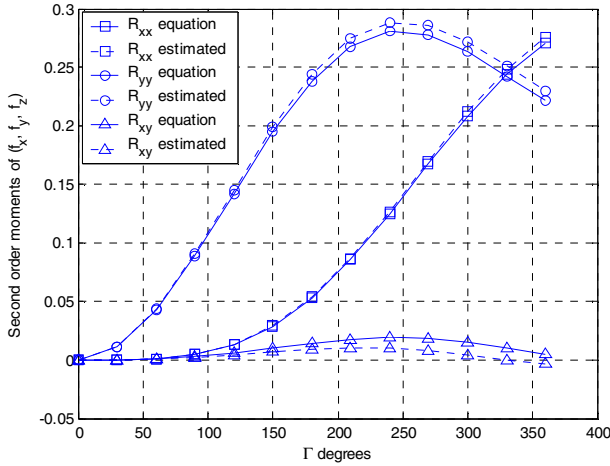


Figure 2: Second order moments of  $(f_x, f_y, f_z)$  against cluster azimuth width, for cluster mean direction  $\mathbf{k} = [1, 0, 0]$  and elevation width  $\Delta = 20^\circ$ . Dotted lines show estimates using (4) and solid lines show results from (9) and (10).

The second order fading moments are a function of the mean cluster angle as well as the angular spread of the cluster. When the mean cluster direction is parallel to the x-axis,  $R_{yy}$  is larger than  $R_{xx}$  due to the greater separation between the directional vectors along the y-axis. As the cluster is rotated in the x-y plane, the second order fading moments are periodic as a function of the mean cluster angle, as shown in Figure 3.

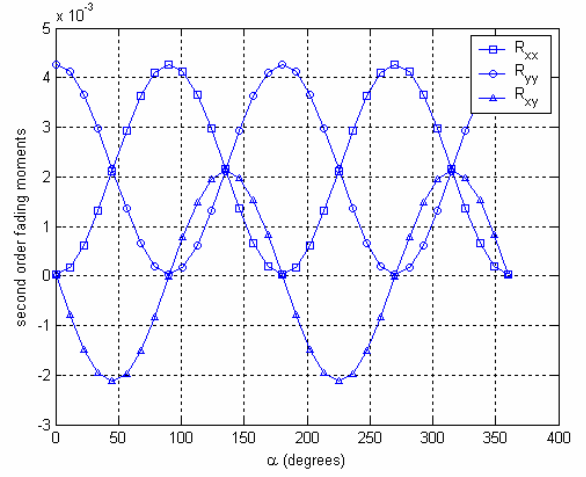


Figure 3:  $R_{xx}$ ,  $R_{yy}$  and  $R_{xy}$  against mean cluster angle in azimuth.

The average  $R_{xx}$ ,  $R_{yy}$  or  $R_{zz}$  of a uni-cluster multipath distribution increases with cluster angular width, as shown in Figure 4. This is expected as  $R_{xx}$  is equivalent to  $\sigma_m^2(\mathbf{u})$  for  $\mathbf{u} = [1, 0, 0]$ .

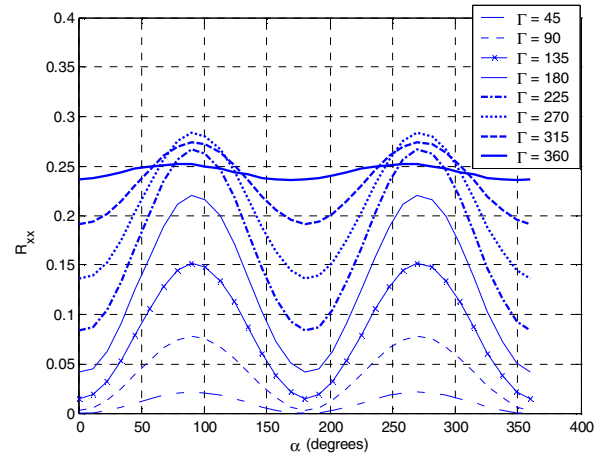


Figure 4:  $R_{xx}$  against cluster mean angle in the x-y plane, for a number of cluster azimuth widths and elevation width of  $\Delta = 20^\circ$ .

2) *Variance of Directional Derivative -  $\sigma_m^2(\hat{\mathbf{u}})$* : For a given directional energy profile  $p(\theta, \phi)$ ,  $\sigma_m^2(\hat{\mathbf{u}})$  gives a measure of decorrelation between elements of a ULA as a function of its orientation, where the line of the array is given by  $\mathbf{u}$ . Figure 5 shows for the case of a single dominant cluster that  $\sigma_m^2(\hat{\mathbf{u}})$  will be maximum when the angular separation between the mean cluster direction and  $\mathbf{u}$  is  $90^\circ$ . This is in agreement with experimental studies where higher capacities were observed when the broadside of a linear array was perpendicular to the direction of the LOS component [16][17]. This also explains why the  $\text{tr}(\mathbf{R})$  (or RMS angular spread) on its own is insufficient, as correlation of a space-diversity array varies with its orientation relative to the direction of the cluster.

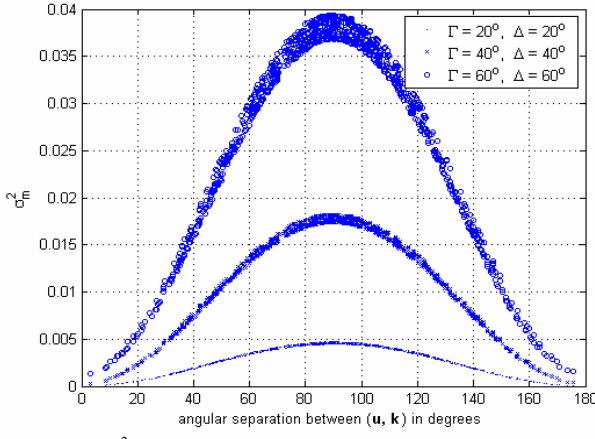


Figure 5.  $\sigma_m^2(\mathbf{u})$  against angular separation of vectors  $\mathbf{u}$  and  $\mathbf{k}$ , for different angular spreads of a cluster.

Now consider the case where multipath energy arrives in two identically distributed clusters from different directions. Here,  $R_{xx}$ ,  $R_{yy}$  and  $R_{zz}$  vary periodically with the overall rotation of the clusters, similar to that shown for the single cluster case in Figure 3.  $R_{xx}$  approaches its maximum value of 0.5 when two clusters are incident from opposite ends of the x-axis and the cluster widths tend towards zero. The general increase in  $\sigma_m^2(\mathbf{u})$  with cluster angular separation is shown for two bi-cluster distributions in Figure 6. At  $\beta = \pi$ ,  $\sigma_m^2(\mathbf{u})$  is maximum when  $\mathbf{u}$  is parallel to the line of the opposite clusters and minimum when it is perpendicular. Here,  $\sigma_m^2(\mathbf{u})$  approaches its upper bound of 0.5 as the constituent cluster angular widths tend to zero. Note that this is in contrast to the single cluster case shown in Figure 5, where the direction of maximum fading is orthogonal to the mean cluster angle. A comparison between Figure 6(a) and Figure 6(b) demonstrates that the variation in  $\sigma_m^2(\mathbf{u})$  with  $\mathbf{u}$  decreases as the constituent cluster angular spreads are increased.

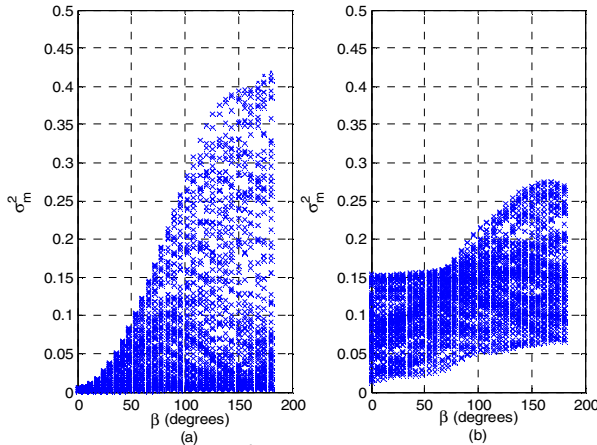


Figure 6: Distribution of  $\sigma_m^2(\mathbf{u})$  against cluster angular separation for a bi-cluster distribution, for random and uniformly distributed directions  $\mathbf{u}$ , for (a)  $\Gamma_1 = \Gamma_2 = 20^\circ, \Delta_1 = \Delta_2 = 20^\circ$ , and (b)  $\Gamma_1 = \Gamma_2 = 120^\circ, \Delta_1 = \Delta_2 = 120^\circ$ .

3) *Directional Spread*: The dependence of  $\text{tr}(\mathbf{R})$  and  $\text{det}(\mathbf{R})$  on the joint azimuth and elevation angular spreads are shown in Figure 7(a) and Figure 7(b) respectively. The results demonstrate that larger azimuth and elevation widths of a

cluster lead to greater average fading over all directions, as given by  $\text{tr}(\mathbf{R})$ , as well as lower variation in second order fading statistics with direction, as given by  $\text{det}(\mathbf{R})$ .

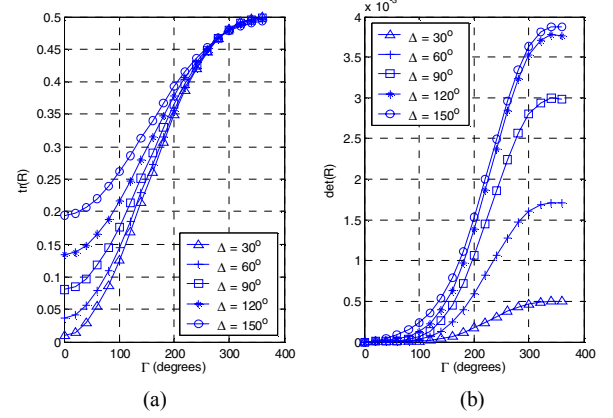


Figure 7.  $\text{tr}(\mathbf{R})$  and  $\text{det}(\mathbf{R})$  of a cluster.

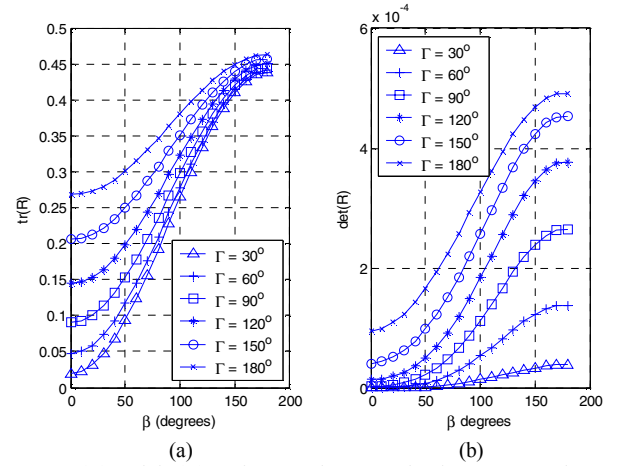


Figure 8.  $\text{tr}(\mathbf{R})$  and  $\text{det}(\mathbf{R})$  against angular separation between two clusters, for  $\Delta_1 = \Delta_2 = 30^\circ$ .

For the two cluster case,  $\text{tr}(\mathbf{R})$  achieves its maximum value of 0.5 when  $\beta = \pi$  regardless of angular widths of the clusters (see Figure 8(a)). However,  $\text{det}(\mathbf{R})$  continues to improve with the cluster angular spreads, as shown in Figure 8(b). This further highlights the importance of  $\text{det}(\mathbf{R})$  as a property of the directional dispersion.

#### APPENDIX A.

A derivation of the general expression for the second order moment of the partial derivatives is shown here. The mean of  $f_x$  (or  $f_y, f_z$ ) over a large number of phase realizations is equal to zero. Using (4) and (8),  $R_{xy}$  can be expressed as (17).

$$R_{xy} = E \left[ \sum_{s=1}^{N_s} \sum_{k=1}^s A_s A_k (d_{x_s} - d_{x_k}) \sin(\psi_s - \psi_k) \sum_{i=1}^{N_s} \sum_{j=1}^i A_i A_j (d_{y_i} - d_{y_j}) \sin(\psi_i - \psi_j) \right] \\ = E \left[ \sum_{s=1}^{N_s} \sum_{k=1}^s \sum_{i=1}^{N_s} \sum_{j=1}^i A_s A_k A_i A_j (d_{x_s} - d_{x_k}) (d_{y_i} - d_{y_j}) \sin(\psi_s - \psi_k) \sin(\psi_i - \psi_j) \right] \quad (17)$$

Note that the amplitudes and directions are treated as constants, and the phases as random variables. The above

mean can be assumed to be zero for summation terms with  $(s,k) \neq (i,j)$  and will be computed only for  $(s,k) = (i,j)$ , as shown in (18).

$$R_{xy} = E \left[ \sum_{s=1}^{N_s} \sum_{k=1}^s A_s^2 A_k^2 (d_{x_s} - d_{x_k}) (d_{y_s} - d_{y_k}) \sin^2(\psi_s - \psi_k) \right] \quad (18)$$

$$= \sum_{s=1}^{N_s} \sum_{k=1}^s A_s^2 A_k^2 (d_{x_s} - d_{x_k}) (d_{y_i} - d_{y_j}) E[\sin^2(\psi_s - \psi_k)]$$

Without loss of generality, the second order moment of fading derivative can be described at a point  $\mathbf{s} = [0, 0, 0]$ . The phase term  $\psi_s$  simplifies to just the random phase component  $\chi_s$  (see (5)). Assuming  $\chi_s$  to be random and uniformly distributed in  $[0, 2\pi]$ , the variance term is given by

$$E[\sin^2(\psi_s - \psi_k)] = E[\sin^2(\phi)] = \int_0^{2\pi} \sin^2(\phi) d\phi = \frac{1}{2} \quad (19)$$

where the random variable  $\phi$  can be expected to be uniformly distributed in  $[0, 2\pi]$ . Using (19), the result in (18) further simplifies to that in (20).

$$R_{xy} = \frac{1}{2} \sum_{s=1}^{N_s} \sum_{k=1}^s A_s^2 A_k^2 (d_{x_s} - d_{x_k}) (d_{y_s} - d_{y_k}) \quad (20)$$

The equivalence between the expressions for  $R_{xy}$  given in (20) and (9) is demonstrated by the fact that they both simplify to the same result, as given by (21) and (22) respectively.

$$R_{xy} = \frac{1}{2} \sum_{s=1}^{N_s} \sum_{k=1}^s A_s^2 A_k^2 (d_{x_s} - d_{x_k}) (d_{y_s} - d_{y_k})$$

$$= \frac{1}{4} \sum_{s=1}^{N_s} \sum_{k=1}^{N_s} A_s^2 A_k^2 (d_{x_s} - d_{x_k}) (d_{y_s} - d_{y_k}) \quad (21)$$

$$= \frac{1}{4} \left[ \sum_{s=1}^{N_s} A_s^2 d_{x_s} \sum_{k=1}^{N_s} A_k^2 d_{y_k} + \sum_{s=1}^{N_s} A_s^2 d_{y_s} \sum_{k=1}^{N_s} A_k^2 d_{x_k} \right.$$

$$\left. - \sum_{s=1}^{N_s} A_s^2 d_{x_s} d_{y_s} \sum_{k=1}^{N_s} A_k^2 - \sum_{s=1}^{N_s} A_s^2 \sum_{k=1}^{N_s} A_k^2 d_{x_k} d_{y_k} \right]$$

$$= \frac{1}{2} \bar{d}_x \bar{d}_y - \frac{1}{2} \sum_{s=1}^{N_s} A_s^2 d_{x_s} d_{y_s}$$

$$R_{xy} = \frac{1}{2} \sum_{s=1}^{N_s} A_s^2 (d_{x_s} - \bar{d}_x) (d_{y_s} - \bar{d}_y)$$

$$= \frac{1}{2} \sum_{s=1}^{N_s} A_s^2 d_{x_s} d_{y_s} + \frac{1}{2} \sum_{s=1}^{N_s} A_s^2 \left( \sum_{k=1}^{N_s} A_k^2 d_{x_k} \right) \left( \sum_{k=1}^{N_s} A_k^2 d_{y_k} \right) + \quad (22)$$

$$- \frac{1}{2} \sum_{s=1}^{N_s} A_s^2 d_{x_s} \sum_{k=1}^{N_s} A_k^2 d_{y_k} - \frac{1}{2} \sum_{s=1}^{N_s} A_s^2 d_{y_s} \sum_{k=1}^{N_s} A_k^2 d_{x_k}$$

$$= \frac{1}{2} \sum_{s=1}^{N_s} A_s^2 d_{x_s} d_{y_s} + \frac{1}{2} \sum_{k=1}^{N_s} A_k^2 d_{x_k} \sum_{k=1}^{N_s} A_k^2 d_{y_k} \sum_{s=1}^{N_s} A_s^2 - \bar{d}_x \bar{d}_y$$

$$= \frac{1}{2} \sum_{s=1}^{N_s} A_s^2 d_{x_s} d_{y_s} - \frac{1}{2} \bar{d}_x \bar{d}_y$$

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